Implementing Level Set Methods in a 2D Boussinesq Model

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Abstract

Tracking surfaces is a challenging issue for a realistic simulation of fluids in interaction with other fluids or solid objects. Level set methods is the state of the art methodology to discretize the movement of a surface in any kind of flow field, by representing it as a scalar variable that is advected by numerical schemes similar to the advection of passive tracers. Here, we apply level set methods to a Boussinesq model, in order to simulate air and water as it moves around its horizontally stratified equilibrium position. The numerical model tends to mix air and water in the presence of shear instability and resulting intermediate densities are unrealistic for air and water. In order to track the water surface, a level set function is implemented. To reduce unrealistic mixing a relaxation approach is proposed: Densities that differ from that of air and water are slightly but consequently restored back to its reference value. Challenges remain in the optimization of this method: A weak relaxation hardly reduces mixing, whereas strong relaxation tends to result in unphysical behaviour.

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1 Introduction

Air and water are two fluids that are, although completely different from a chemical point of view, physically fairly similiar. Neglecting viscosity and compressibility, they only differ in their density for physical purposes. By scaling analysis, we may find the same phenomenon governing the fluids motion in the ocean as well as in the atmosphere, although on different time and length scales. Hence, investigating fluid dynamics often means to treat the fluid as if it could represent air or water interchangeably. However, the physics that appear while studying the interaction between both fluids do not arise from the physical models we are used to investigate while looking at these fluids seperately. Instead, the surface between air and water is largely dominated by effects such as the inability to mix: From a chemical point of view air and water may be found in a heterogenous mixture but rather not in a homogeneous one (apart from tiny amounts of oxygen that is solved in water). Hence, the simulation of propagating surfaces need to involve more than just equations of motions as known for the fluids seperately.

Here, we present level set methods as developed by others mainly for the purpose of movie rendering, for realistic interactions between fluids and/or solids. Level set methods may also be used as numerical technique to implement unresolved mechanisms such as surface tension. We apply level set methods in order to track the surface between air and water in a simulated 2D tank including gravity. Starting point is, as can be seen in Figure 1 a non-hydrostatic but Boussinesq model, that involves mixing, as arising from the numerical schemes. We use level set methods and a relaxation approach

to reduce the mixing in the model. Section 2 defines the mathematical background for level set methods closely following previous work and section 3 its numerical implementation. The relaxation approach is presented in section 4 in detail. Section 5 presents some results from implementing level set methods and relaxation into the model for one experiment with the model. Section 6 draws a conclusion and discusses the results and finally, section 7 provides some perspectives for future work.

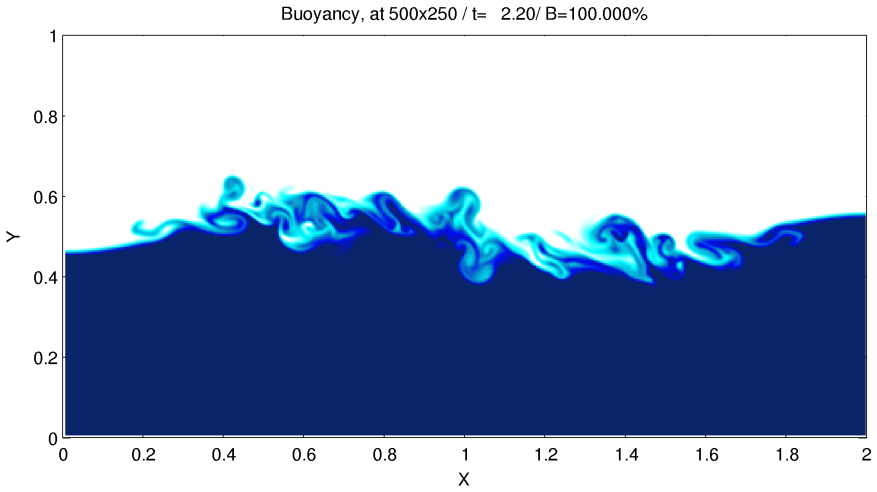

Figure 1: Mixing of two fluids with different density as arising from shear instability in the Boussinesq model. Mixing occurs for fluids such as fresh water and sea water but not for air and water due to chemical processes.

2 Mathematical definition and properties

In this section the notation is mainly adopted from [1], please see [2] or [4] for similiar approaches of defining the level set function.

In order to follow the movement of water in a 2D tank we may regard its area as a set Ω, so that all points $\mathbf{x} = (x, z)$ in a cartesian x,z-coordinate system that satisfy $\mathbf{x} \in \Omega$ are considered to be water. All points $\mathbf{x}$ with $\mathbf{x} \notin \Omega$ are thus air.

We aim to track the *interface* Γ, i.e. the surface between air and water, as the fluids move around in the tank. The interface Γ is mathematically the boundary of Ω,

$$\Gamma = \partial\Omega. \tag{1}$$

By defining now the *level set function* Φ, with properties such as

$$\Phi(\mathbf{x}, t) \begin{cases} > 0 & \text{if } \mathbf{x} \in \Omega \\ < 0 & \text{if } \mathbf{x} \notin \Omega \\ = 0 & \text{if } \mathbf{x} \in \partial\Omega = \Gamma, \end{cases} \tag{2}$$

we obtain a scalar quantity that is defined for every point $\mathbf{x}$ of our domain (i.e. tank). See Figure 2 as a schematic example. We want the level set function

Φ to follow at any time t the flow field $\mathbf{u}(\mathbf{x},t) = (u(x,z,t), w(x,z,t))$, hence its value shall not change along a trajectory. From an Eulerian point of view we thus have

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \nabla \Phi = 0. \tag{3}$$

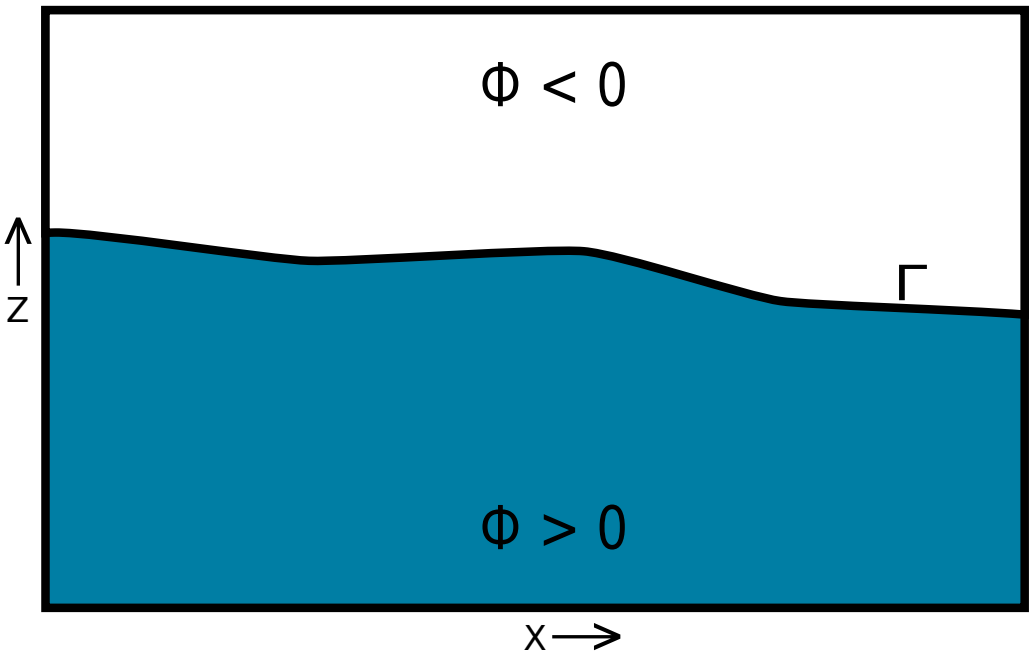

Figure 2: 2D tank with water and air. The level set function Φ is defined to be positive for water and negative for air. The interface Γ is located where $\Phi = 0$.

2.1 Distance function

So far only the sign of the level set function is defined. We now further specify its absolute value, by regarding Φ as a *distance function* that still satisfies equation (2) and includes the information of the distance d from any point $\mathbf{x}$ in the domain to the interface Γ. So that Φ becomes

$$\Phi(\mathbf{x},t) = \pm d \tag{4}$$

with the sign of d to chosen positive for water and negative for air. For practical purposes it is not necessary to define the distance far away from the interface, but still defining the level set function actually as distance function comes along with a control of the interface thickness γ that is numerically on the order of a few grid points.

3 Numerical implementation

The general procedure is now as follows: From equation (2) and (4) we set up a new state variable corresponding to our level set function, which is then

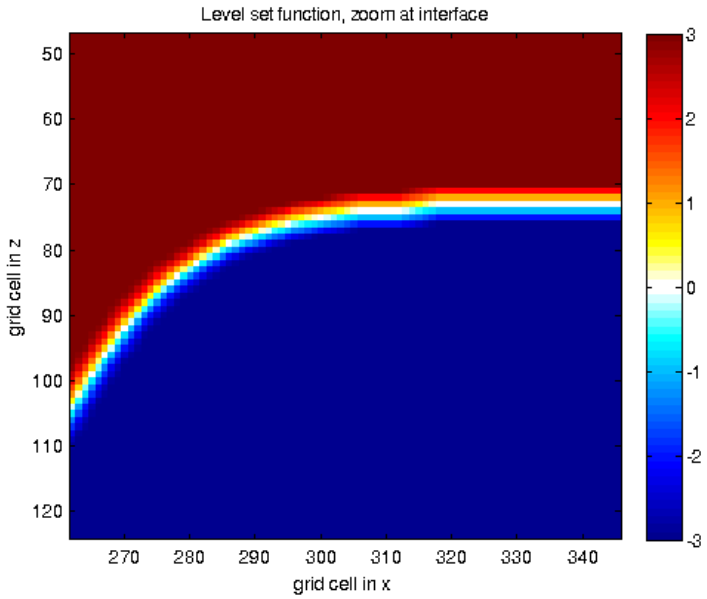

Figure 3: The level set function is set up as distance function. The distance from the interface is measured in grid cells with a maximum value of half the interface thickness γ. Here $\gamma = 6$.

on every time step advected by the model variable $\mathbf{u}$ as described in equation (3) in order to follow the density surface. The zero level set of $\Phi(\mathbf{x}, t)$ is then regarded as the propagating interface $\Gamma(t)$.

3.1 Initialization of the level set function

We start our model from rest, i.e. at $t = 0$ the following variables vanish

$$\mathbf{u}(t = 0) = \omega(t = 0) = 0 \tag{5}$$

with ω being the vorticity. Instead, we set a certain density $\rho(\mathbf{x}, t = 0) = \rho_0$ as initial conditions. We deduce ρ_0 from an initial interface $\Gamma_0 = \Gamma(t = 0)$ and its corresponding level set function Φ_0. Γ_0 should be smooth and appropriate to represent a water surface. The density ρ_0 is then

$$\rho_0 = \begin{cases} \rho_{water} & \text{for } \{\mathbf{x} \mid \Phi_0 > \gamma/2\} \\ \rho_{air} & \text{for } \{\mathbf{x} \mid \Phi_0 < \gamma/2\}. \end{cases} \tag{6}$$

For numerical purposes we introduce the interface thickness γ, that is usually measured in grid cells. For $\gamma \to 0$ we encouter numerical problems as the resulting buoyancy torque $\frac{\partial b}{\partial x} = -\frac{\partial \rho}{\partial x}$ goes to very large values. Hence, choosing γ on the order of a few grid cells is appropriate. The density close to the interface should be set up as a smooth change from ρ_{air} to ρ_{water}.

3.2 Initialization of the distance function

Instead of setting up a level set function with arbitrary values positive values for water and negative for air, we let the distance function be our level set

function, which defintion follows then from equation (4). Although a distance of a point to a curve might be mathematically well defined, numerically this solution is not trivial and different solutions with varying computing time exist. Here we propose the following:

Let the level set function temporarily be initialized as follows

$$\tilde{\Phi_0} = \begin{cases} \frac{\gamma}{2} & \text{for } \mathbf{x} \in \Omega, \\ -\frac{\gamma}{2} & \text{for } \mathbf{x} \notin \Omega. \end{cases} \tag{7}$$

The main idea is now to smooth $\tilde{\Phi_0}$ by averaging with its neighbours. Hence, we apply the following $\gamma \times \gamma$ stencil on every grid point of $\tilde{\Phi_0}$

$$\frac{1}{\gamma^2} \cdot \begin{matrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{matrix} \tag{8}$$

For every $\mathbf{x}$ at least $\gamma/2$ away from the interface this does not change the value of $\tilde{\Phi_0}$ there, so that this computation may be skipped to reduce the computing time. For a grid cell that is located on Γ_0, we average as many grid cells with positive values as with negative and hence obtain our zero-level set at the interface. This is exact in the case where the curvature of Γ_0 vanishes but still a good approximation in other cases. In fact, applying the stencil from (8) on $\tilde{\Phi_0}$ yields Φ_0 an approximation of the distance function, that can be used as level set function.

An Example of this discrete distance function is given in Figure 3. We see that the interface thickness γ is largely maintained also for curved parts of Γ.

4 Relaxation approach to reduce mixing

The property whether two fluids mix arises from chemical processes. Usually the advective schemes of a numerical model involve mixing, hence we seek here to find a method that reduces the effect of mixing.

4.1 Restoring to the initial density

We choose a relaxation approach in order to restore the density of our fluid back to its reference value. The level set function is used to choose whether the reference density is that from air or from water. Let $\rho(\mathbf{x}, t)$ be the state vector of density at a given time t, and $\rho(\mathbf{x}, t)_{ref}$ the reference density at that point. Please note that the reference density is time dependent in an Eulerian formulation as the interface moves. For some t a given point $\mathbf{x}$ may represent water, whereas at another t this point might be air. The reference density is found via

$$\rho(\mathbf{x}, t)_{ref} = \begin{cases} \rho_{water} & \text{for all } \{\mathbf{x} \mid \ \Phi(\mathbf{x}, t) > \epsilon\} \\ \rho_{air} & \text{for all } \{\mathbf{x} \mid \ \Phi(\mathbf{x}, t) < -\epsilon\} \\ \text{undefined} & \text{for all } \{\mathbf{x} \mid \ |\Phi(\mathbf{x}, t)| < \epsilon\}. \end{cases} \tag{9}$$

with a small parameter ϵ which represents the threshold thickness, where relaxation is applied. Normally we choose

$$0 < \epsilon < \frac{\gamma}{2}, \tag{10}$$

so that restoring is not applied directly at the interface but at some distance $d > \epsilon$ in order to avoid too steep density gradients. The case where $\epsilon = \gamma/2$ involves no restoring, as $\max(\Phi) = \gamma/2$. Having found the reference density we restore the present density ρ to a corrected density ρ' on every time step by

$$\rho'(\mathbf{x}) = \rho(\mathbf{x}) - \lambda(\mathrm{dt})\left(\rho(\mathbf{x}) - \rho_{ref}(\mathbf{x})\right) \tag{11}$$

with $\lambda(\mathrm{dt})$ the relaxation parameter, that is here given as explicity dependent on the numerical time spacing dt. To obtain a relaxation parameter that is independent on time t, we choose λ to be proportional to dt, by

$$\lambda(\mathrm{dt}) = \lambda_0\,\mathrm{dt}\,. \tag{12}$$

It is obviously necessary to choose

$$0 \leq \lambda(\mathrm{dt}) \leq 1 \tag{13}$$

as negative values correspond to a repulsion and $\lambda > 1$ to an overshooting. $\lambda = 0$ is the case of no relaxation, and $\lambda = 1$ a perfect relaxation that restores the density immediately back to its reference value.

4.2 Re-set up the level set function

Corresponding to equation (3) the numerical model treats the level set function as a tracer and advects exactly in the same way as the density is advected. As discussed, without relaxation the density of air and water starts to mix, as any numerically advected gradient tends to disperse. We choose therefore to re-set up the level set function based on the density, with the following assumption

$$\Gamma = \{\mathbf{x} | \rho(\mathbf{x}) = \bar{\rho}\} \tag{14}$$

with $\bar{\rho} = (\rho_{air} + \rho_{water})/2$. Hence, we assume that the mean value of water and air density is an appropriate representation of the interface Γ. From Γ we then deduce the level set function by calculating the distance as already described in section 3.2.

This procedure is not done every time step. Instead, we identify another parameter m, an integer that quantifies the number of integration steps until a new level set function is set up. The parameter m impacts the resulting physics as it influences the relaxation itself and the performance, as the re-set up is computationally costly.

5 Results

We follow the procedure of the previous sections to obtain the level set function and implement it in the advective scheme of the numerical model. The model is then started from rest with initial conditions as shown in Figure 4. The surface obeys a hyperbolic tangent with a steep step in the middle of the domain, in order to have a strong buoyancy torque that sets the fluids in motion. The level set function shows mainly the values $\pm\gamma/2$ and only next to Γ it is actually representing a distance function. The interface as shown below is rendered as the $0th$ contour line of Φ.

At some later time $t = 0.55$ that is reached after 169 integrations, shear instabilities tend to curl the surface, known as Kelvin-Helmholtz instability, as presented in Figure 5. The level set function follows that movement so that a spiral shaped interface becomes visible.

Even after about 1000 integrations and a complicated flow field, the level set function is still able to track the surface as seen in Figure 6. A closer look reveals that some parts of either fluids deattached and drops are visible. Nevertheless, fine structures as the spiral in the middle of the domain, which is still present in buoyancy is not captured by the interface.

The resulting interface looks fairly different from that obtained without relaxation (Fig. 1). We conclude that the relaxation approach therefore affects the model physics, whereas the details of this influence remain uninvestigated.

5.1 Quantifying mixing

Furthermore, we analyze the effectivity to reduce mixing with means of probability density distributions as shown in Figure 7. The case of applying relaxation is compared to the same simulation without restoring at $t = 5$, i.e. after about 2000 integrations. Obviously reduced are densities between 100 and 950, whereas relaxation also results in a broader peak around $\rho_{air} = 1$ and $\rho_{water} = 1000$, including completely unrealistic negative densities. Since the density enters the model equations only in terms of its torque, negative densities seem to cause less problems than they would do in the real world, as the numerical scheme remains stable.

6 Concluding discussion

We implemented a level set function into an existing 2D model, that integrates the vorticity equation for non-hydrostatic fluids with Boussinesq approximation. The level set function is advected in Eulerian manner, and follows largely the flow field. Especially for fine structures such as spirals and filaments, the level set function seem to lack in accuracy, which mainly arises from a slightly dispersive advective scheme.

A relaxation approach was succesfully tested, although we recognize that its behaviour is still far from optimum. Potential may arise from tuning the level set and relaxation parameters in order to optimize the density restoring. A weaker restoring was observed to be not able to reduce the mixing, whereas

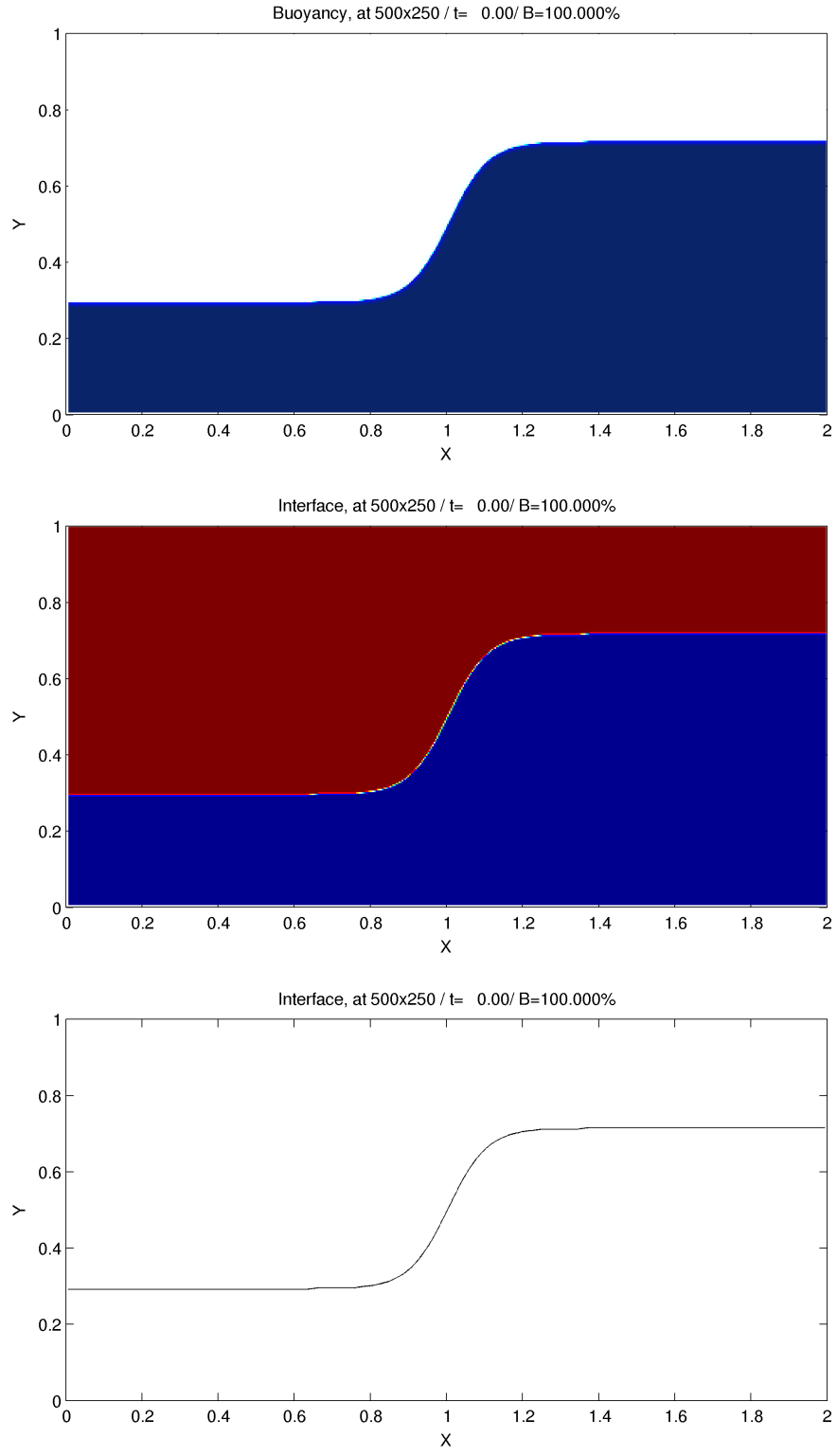

Figure 4: Initial conditions. The model starts from rest, but with a step in density (upper; white: air, blue:water). The initial buoyancy torque will accelerate the fluids and let them swing around the equilibrium position of fully horizontally stratified fluids. (middle) The level set function at $t = 0$. (lower) The intital interface Γ as resulting from $\Gamma = \{\mathbf{x} \mid \Phi(\mathbf{x}, t = 0) = 0\}$.

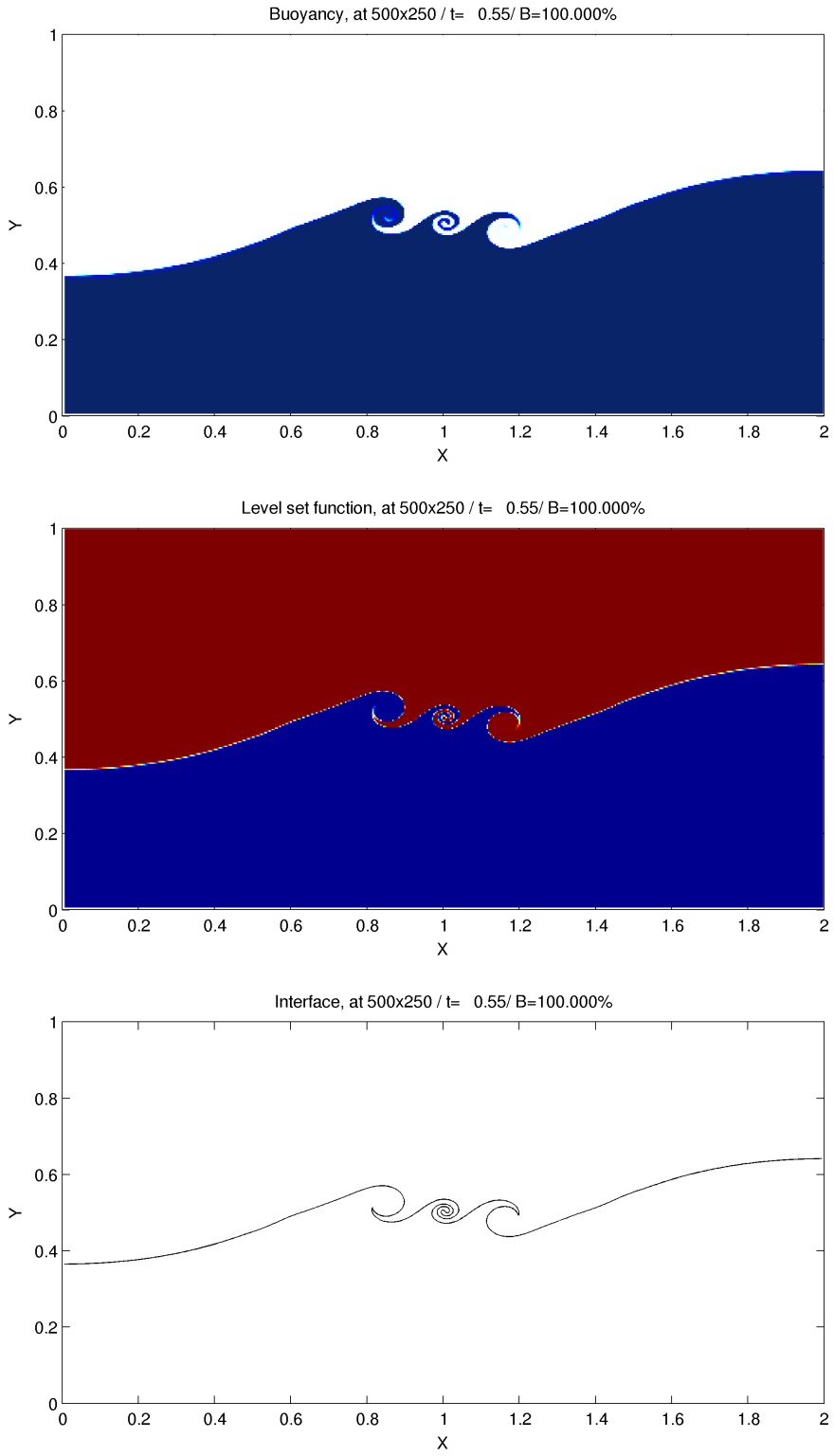

Figure 5: Kelvin-Helmholtz-Instability. Due to the level set function the interface is tracked and the typical spiral shape of Kelvin-Helmholtz instabilites becomes visible.

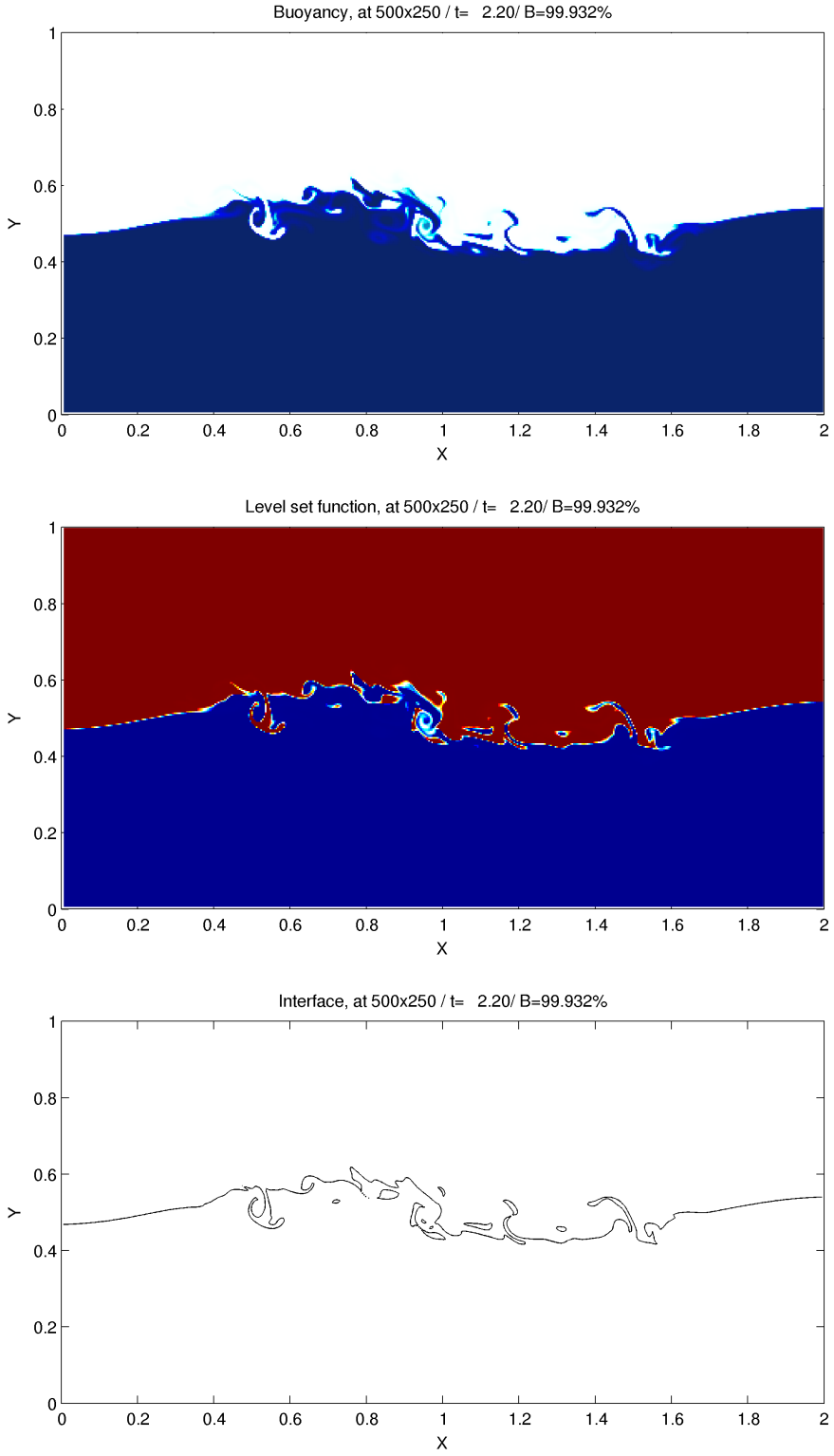

Figure 6: The level set function is able to follow complex interfaces including water drops.

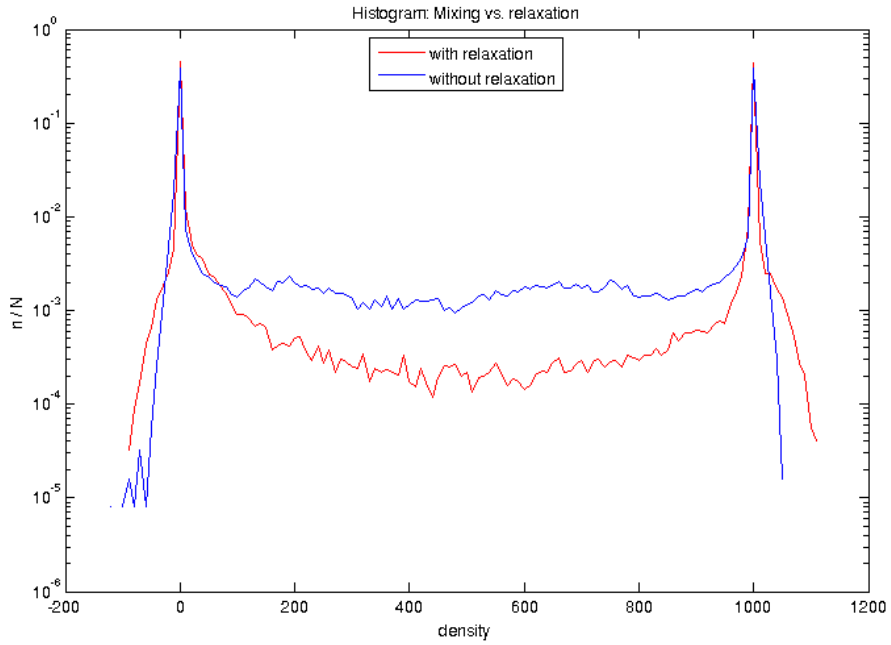

Figure 7: Relaxation reduces the amount of grid cells with undesired intermediate density as resulting from mixing. The histogramm was computed with a bin size of 10 density units.

a stronger restoring was observed to largely change the model physics, as fine structures suddenly disappear.

7 Perspectives

Although some progress was made by using level set methods to reduce mixing, the realistic simulation of surface gravity waves with an air-water interface remains a challenge. The relaxation method as presented here is not optimized, physical features such as surface tension are not implemented yet. Here, we want to provide perspectives for future research in adjusting the model equations to also allow for settings where the Boussinesq assumption does not hold.

7.1 The non-Boussinesq case

In the case of air and water, we know that waves that travel at the interface may become unstable, as for swell at the beach. They increase in height, bow forward and break, producing turbulence that transports their energy to small scales where dissipation occurs. However, spiral-like wave breaking as known from Kelvin-Helmholtz instabilities are not observed for air-water interfaces.

We suggest, that this results, as the numerical model is actually solving the non-hydrostatic Euler equations with the Boussinesq approximation, which idea is as follows: Differences in density are assumed to result in a vertical acceleration while multiplied by the gravity acceleration g. In first

order, they do not affect the time derivative on the left hand side, where density ρ_0 is assumed to be constant where not multiplied with g, as follows from the assumption that

$$\frac{\rho_1 - \rho_2}{\rho_2} \ll 1 \qquad \text{or} \qquad \frac{\rho_1}{\rho_2} \approx 1. \tag{15}$$

That means, density differs only slightly in terms of its ratio. The Boussinesq approximation is valid for e.g. internal waves, where density is still on the order of 1000 but may vary as resulting from variation in temperature or salinity. However, the Boussinesq approximation does not hold for air and water, as equation (15) is strongly violated.

In order to simulate therefore realistically surface gravity waves it is necessary to change the model equations by introducing a true momentum $\mathbf{m} = \rho\mathbf{u}$, the non-hydrostatic Euler equations become (G. Roullet, 2014, personal communication)

$$\frac{\partial \mathbf{m}}{\partial t} + \mathcal{J}(\psi, \mathbf{m}) = -\nabla p + \rho\mathbf{g} \tag{16a}$$

$$\frac{\partial \rho}{\partial t} + \mathcal{J}(\psi, \rho) = 0 \tag{16b}$$

with $\mathcal{J}$ the Jacobian, ψ the streamfunction and p the pressure. Taking the curl yields the non-Boussinesq Euler equations in vorticity formulation

$$\frac{\partial \zeta}{\partial t} + \mathcal{J}(\psi, \zeta) = -g\frac{\partial \rho}{\partial x} + \mathcal{J}(\rho, K) \tag{17a}$$

$$\zeta = \nabla \cdot (\rho \nabla \psi) \tag{17b}$$

with $K = \|\mathbf{u}\|^2/2$, the kinetic energy density and the density modified vorticity ζ. Numerical problems arise as the previous Laplacian $\Delta = \nabla \cdot \nabla$ is not uniform anymore, but in sandwich with the density and therefore changing for every integration. Hence, a cholesky factorization is not appropriate.

The behaviour of these modified equations and fast numerical schemes to invert the vorticity for varying Laplcians need to be investigated in future research projects.

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